

Impulsive spherical gravitational waves

A. N. Aliev and Y. Nutku

Feza Gürsey Institute, P. K. 6 Çengelköy 81220 Istanbul, Turkey

November 6, 2000

Penrose's identification with warp provides the general framework for constructing the continuous form of impulsive gravitational wave metrics. We present the 2-component spinor formalism for the derivation of the full family of impulsive spherical gravitational wave metrics which brings out the power in identification with warp and leads to the simplest derivation of exact solutions. These solutions of the Einstein vacuum field equations are obtained by cutting Minkowski space into two pieces along a null cone and re-identifying them with warp which is given by an arbitrary non-linear holomorphic transformation. Using 2-component spinor techniques we construct a new metric describing an impulsive spherical gravitational wave where the vertex of the null cone lies on a world-line with constant acceleration.

1 Introduction

For weak fields and slow motion, the emission of gravitational radiation requires that the source must possess at least a quadrupole moment [1]. The experimental verification of the "quadrupole formula" has been spectacular [2]. However, this radiation is weak, down by a factor of G/c^5 which is an infamous fourty orders of magnitude. On the other hand, an extrapolation of the quadrupole formula to strong fields by dimensional analysis [3] turns this factor around to its inverse and suggests that gravitational radiation is the mechanism for releasing most energy by anything anyhow.

For strong gravitational fields there is a mechanism for the emission of gravitational waves which makes no reference to quadrupole radiation. It starts with an apparent paradox: Monopole radiation is forbidden by the

principle of equivalence, yet we have exact solutions of the Einstein field equations that describe spherical gravitational waves. These are the famous solutions of Robinson and Trautman [4]. Soon after their discovery it was realized that Petrov Type N Robinson-Trautman solutions possess “wire singularities,” that is, the wave-fronts are locally spherical but they are not complete. If we cut a paraboloid out of Minkowski space, impose perfectly reflecting boundary conditions and shine an impulsive plane gravitational wave [5] on it, then an impulsive spherical wave will form in the neighborhood of the focus. The paraboloid must be of finite extent for the spacetime to support the impulsive plane wave and therefore a cone with the opening angle of the paraboloid will be missing out of the wave-front. Thus a contradiction with the principle of equivalence is avoided but such a construction sheds little light on the physical origin of Type N Robinson-Trautman metrics.

Considerations of topological defects which may have formed in the early universe [6] provide an interesting possibility for the physical interpretation of wire singularities in radiative Robinson-Trautman metrics. Indeed, cosmic strings which are physically the most feasible topological defects have proved to be very popular for restoring to physics all sorts of metrics with conical singularities. However, an examination of the properties of continuous metric for an impulsive spherical wave (1) shows that nowhere is the case for cosmic strings as compelling as it is for impulsive spherical gravitational waves where they have an unambiguous role to play: *a snapping cosmic string emits impulsive spherical gravitational waves and is thereby annihilated.*

The exact solution describing an impulsive spherical gravitational wave cannot be given in the Robinson-Trautman form because of severe discontinuities encountered in the Robinson-Trautman coordinate patch. The continuous form of the metric is obtained by the “scissors and paste” approach that Penrose [7], [8] has devised for the study of impulsive gravitational waves. Impulsive wave metrics are obtained by cutting Minkowski space into two pieces along a null hypersurface and gluing them back together after warp. For impulsive spherical gravitational waves the null hypersurface is a cone and Penrose’s identification with warp yields the metric [9]

$$ds^2 = 2 du dv - 2 \left| u d\bar{\zeta} + v \theta(v) \{h; \zeta\} d\zeta \right|^2 \quad (1)$$

where θ is the Heaviside unit step function, h is an arbitrary holomorphic

function

$$h_{\bar{\zeta}} = 0 \quad (2)$$

which determines the warp and

$$\{h; \zeta\} = -\frac{1}{2} \left(\frac{h'''}{h'} - \frac{3}{2} \frac{h''^2}{h'^2} \right) \quad (3)$$

is its Schwarzian derivative. This metric satisfies Einstein's vacuum field equations for all v and it is continuous across the null hypersurface $v = 0$. It is Petrov Type N with the Weyl curvature suffering a Dirac δ -function discontinuity at $v = 0$ that establishes the impulsive character of the spherical wave. The metric (1) remained unpublished until 1992 even though quantum effects in its background were studied by Hortaçsu [10] which marked its first appearance in print. Meanwhile Gleiser and Pullin [11] have independently found a special case of eq.(1) which results from the choice of an exponential for the warp function and gave its correct interpretation as the radiation accompanying the snapping of a straight, infinitely long cosmic string. Later, Hogan has repeated the derivation of the metric (1) for an arbitrary holomorphic warp function. Griffiths and Podolský [12], [13] have presented exact solutions for impulsive spherical waves in de Sitter and anti-de Sitter spacetimes. They have also discussed the relationship of impulsive spherical wave solutions to collision and snapping of cosmic strings [14]. A different approach to the collision problem was proposed by Tod [15]. We shall not discuss the collision problem. In our case the physical interpretation of the impulsive spherical wave metrics requires a cosmic string that has already snapped and is acting as the source of the radiation. This is based on the fact that Minkowski space outside the light cone has a conical deficit whereas inside it is complete.

In this paper we shall present 2-component spinor techniques to construct an arbitrary holomorphic transformation of Minkowski space which is the proper framework for constructing impulsive spherical wave metrics. This is a point emphasized by Penrose [8]. Penrose's identification with warp imposes continuity conditions on the spinor field defining the null frame on both sides of Minkowski space after the removal of the null cone. We shall also discuss physical properties of this metric and show that in the exterior region of the future null cone $v > 0$ the metric exhibits conical singularities characteristic of a cosmic string. This means that the two regions of Minkowski space $v < 0$

and $v > 0$ are matched at $v = 0$ and the full spacetime obtained by the union of these three pieces describes a cosmic string which has already snapped and its free ends are moving away at the speed of light. The impulsive spherical gravitational wave is at the lightcone and flat spacetime without conical deficit is left behind. Thus the cosmic string is getting annihilated as the impulsive spherical gravitational wave propagates.

We shall present a new Type N exact solution of the Einstein field equations describing an impulsive spherical gravitational wave that incorporates uniform acceleration into the solution. Petrov Type N Robinson-Trautman spherical gravitational wave metrics contain a parameter $k = 0, \pm 1$ which can be regarded as specifying the world-line of a corresponding source for spherical gravitational waves and in identification with warp it coincides with the world-line of the vertex of the null cone. The metric (1) corresponds to the null, $k = 0$, case. We shall show that it is also possible to choose a world-line subject to constant acceleration. This is possible only for non-vanishing k . We shall therefore start by applying Penrose's identification with warp to obtain the impulsive spherical wave metric for all values of the parameter k which reduces to Hogan's solution [17] for $k = +1$. Then we shall present the new solution by constructing the arbitrary holomorphic transformation of Minkowski space for the case of a constant acceleration world-line.

2 Penrose's identification with warp

We recall [8] that the construction of an impulsive spherical wave through Penrose's scissors and paste approach starts with the removal of a null cone \mathcal{N} from Minkowski space. This leaves two disjoint Minkowskian regions and the metrics appropriate to these regions are derived from the standard Minkowski interval

$$ds^2 = 2 dU' dV' - 2 dZ' d\bar{Z}' \quad (4)$$

by means of the transformation

$$\begin{aligned} U' &= u, \\ V' &= v + u |\zeta|^2, \\ Z' &= u \zeta \end{aligned} \quad (5)$$

so that the metric of Minkowskian region \mathbf{M} outside a future null cone \mathcal{N} ($v > 0$) is given by

$$ds^2 = 2 du dv - 2u^2 d\zeta d\bar{\zeta}, \quad (6)$$

while another Minkowskian region $\hat{\mathbf{M}}$ inside the cone \mathcal{N} ($v < 0$) is described by the metric

$$d\hat{s}^2 = 2 d\hat{u} d\hat{v} - 2\hat{u}^2 d\hat{\zeta} d\hat{\bar{\zeta}}. \quad (7)$$

Here v is a null coordinate, u is a Bondi-type luminosity distance and ζ is the stereographic coordinate on the Riemann sphere. In Penrose's identification with warp these two disjoint regions are reattached along a null cone \mathcal{N} according to the relations

$$\begin{aligned} \hat{v} &= 0 = v, \\ \hat{u} &= \frac{u}{|h'|}, \\ \hat{\zeta} &= h(\zeta) \end{aligned} \quad (8)$$

where we recall that h is an arbitrary holomorphic function and prime denotes derivative with respect to its argument. The (u, ζ) -transformation in eqs.(8) is an arbitrary holomorphic transformation of the spin-space at the vertex of a future null cone \mathcal{N} . In order to show this we choose at each point of the null cone \mathcal{N} a 2-component spinor ξ^A such that the position vector of a point on the cone can be expressed as

$$x^\mu \leftrightarrow x^{AX'} = u \xi^A \xi^{X'}$$

where in terms of the coordinates (5) we can express this by the trace-free spin matrix

$$\begin{pmatrix} 1 & \bar{\zeta} \\ \zeta & \zeta\bar{\zeta} \end{pmatrix} = \begin{pmatrix} \xi^0 \bar{\xi}^{0'} & \xi^0 \bar{\xi}^{1'} \\ \xi^1 \bar{\xi}^{0'} & \xi^1 \bar{\xi}^{1'} \end{pmatrix} \quad (9)$$

and the 2-component spinor is given by

$$\xi^A = \begin{pmatrix} 1 \\ \zeta \end{pmatrix}. \quad (10)$$

Penrose's requirement of Type *I* geometry is expressed by the invariance of the 1-form

$$\alpha = \xi_A d\xi^A \quad (11)$$

under identification with warp. Explicitly, an arbitrary holomorphic mapping $\zeta \rightarrow h(\zeta)$ preserving the differential form (11) on the future null cone \mathcal{N} corresponds to the 2-component spinor of the form

$$\hat{\xi}^A = \begin{pmatrix} h \\ \zeta \eta^{1/2} \end{pmatrix} \quad (12)$$

where

$$\eta = \frac{1}{h'} \quad (13)$$

which yields the last one of eqs.(8) in the identification with warp. The Hamiltonian nature of the twistor transformation between Minkowski spaces \mathbf{M} and $\hat{\mathbf{M}}$ [16] is related to the invariance of the 1-form (11)

$$\xi_A d\xi^A = d\zeta \quad (14)$$

which implies that the symplectic 2-form

$$\omega = d\xi_A \wedge d\xi^A \quad (15)$$

is also invariant under identification with warp.

3 Holomorphic transformation

So far we have dealt with conditions of Penrose's Type *I* geometry which insures continuity of the induced degenerate metric on \mathcal{N} . This is a requirement restricted to the surface $v = 0$. But the construction of the continuous metric on the full space $\mathcal{M} = \mathbf{M} \cup \mathcal{N} \cup \hat{\mathbf{M}}$ that describes impulsive spherical gravitational waves requires information in addition to eqs.(8). Now we must demand that the conditions for Penrose's Type *II* and *III* geometry must be satisfied as well. We start with continuity of the second fundamental form. For this purpose we shall introduce another 2-component spinor μ^A with flagpole direction along a straight null-path. Then the 2-component spinor ξ^A will have flagpole direction along a null cone with vertex on the straight null-path. For the position vector of a general point we have therefore

$$\begin{aligned} x^\mu \leftrightarrow x^{AX'} &= u \xi^A \xi^{X'} + v \mu^A \mu^{X'} \\ &= u \begin{pmatrix} 1 & \bar{\zeta} \\ \zeta & \zeta \bar{\zeta} \end{pmatrix} + v \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \quad (16)$$

from which one can read off the constant 2-component spinor

$$\mu^A = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (17)$$

satisfying the normalization condition

$$\xi_A \mu^A = 1. \quad (18)$$

Now performing an arbitrary holomorphic transformation $\zeta \rightarrow h(\zeta)$ we can present the position vector of a general point in the form

$$\hat{x}^{AX'} = u|\eta| \begin{pmatrix} 1 & \bar{h} \\ h & |h|^2 \end{pmatrix} + v|\tilde{\eta}| \begin{pmatrix} 1 & \bar{m} \\ m & |m|^2 \end{pmatrix} \quad (19)$$

and the 2-component spinor μ^A can be written in a form analogous to eqs.(12)

$$\hat{\mu}^A = \begin{pmatrix} \tilde{\eta}^{1/2} \\ m \tilde{\eta}^{1/2} \end{pmatrix} \quad (20)$$

where the two unknown holomorphic functions m and $\tilde{\eta}$ have to be determined according to the requirements of Penrose's Type *II* and *III* geometry. That is, the normalization condition (18) and

$$\mu_A \frac{d\xi^A}{d\zeta} = 0 = \hat{\mu}_A \frac{d\hat{\xi}^A}{d\zeta} \quad (21)$$

must be invariant under the holomorphic warp. From eqs.(18) and (21) we find

$$m = h - 2 \frac{h'^2}{h''} \quad (22)$$

and

$$\tilde{\eta} = \frac{h'}{(m - h)^2} \quad (23)$$

which completes the determination of the 2-component spinor $\hat{\mu}^A$. Substituting the expressions (13), (22) and (23) into eq.(19) we find the transformation to the flat region $v > 0$ behind the impulsive wave

$$U' = \frac{u}{|h'|} + \frac{v}{4|h'|} \left| \frac{h''}{h'} \right|^2,$$

$$\begin{aligned}
V' &= \frac{u}{|h'|} |h|^2 + \frac{v}{4|h'|} \left| \frac{h''}{h'} \right|^2 |m|^2, \\
Z' &= \frac{u}{|h'|} h + \frac{v}{4|h'|} \left| \frac{h''}{h'} \right|^2 m
\end{aligned} \tag{24}$$

which highlights the role played by the spinors ξ^A and μ^A . With this transformation the metric (1) is obtained readily. It is a continuous metric for all values of v which is a Petrov Type N exact solution Einstein's vacuum field equations.

3.1 Properties

It will be useful to recall some properties of the Schwarzian derivative [18] because they play an important role in the physical interpretation of the metric (1) as the radiation accompanying the snapping of a cosmic string. First of all we have $PGL(2, C)$ -invariance

$$\left\{ \frac{\alpha h + \beta}{\gamma h + \delta}; \zeta \right\} = \{h; \zeta\} \quad \forall \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL(2, C) \tag{25}$$

which shows that the metric (1) is one of the simplest examples in twistor theory. Using the connection formula where under an arbitrary analytic change $\zeta \rightarrow z$ we have

$$\{h; z\} dz^2 = \{\zeta; z\} dz^2 + \{h; \zeta\} d\zeta^2, \tag{26}$$

that is, $h(\zeta)$ is $PGL(2, C)$ -multi-valued if any of its two branches are projectively related, but $\{h; \zeta\}$ is single-valued. It is the single-valued Schwarzian derivative that appears in the metric. Further, if we consider the null co-frame

$$l = dv, \quad n = du, \quad m = u d\bar{W}_1 + v\theta(v) dW_2 \tag{27}$$

where dW_1 and dW_2 are Abelian differentials, then the condition for the quadratic form $dW_1 dW_2$ to be Schwarz-integrable is

$$dW_1 dW_2 = \{h; \zeta\} d\zeta^2 \tag{28}$$

which is precisely the form realized in the metric (1).

The phase of a spinor is defined by the stereographic projection of the intersection of the null cone with the unit sphere. It follows from eq.(10) that it is the phase that undergoes the warp. If the warp were to be given by a Möbius transformation

$$h = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta} \quad (29)$$

which is simply a uniform rotation of the Riemann sphere, then it would not be a true warp at all and there will be no impulsive spherical gravitational wave. This is confirmed by the vanishing of the Schwarzian derivative for h given by eq.(29).

The relationship between the impulsive spherical gravitational wave and snapping cosmic string is related to the local behaviour of the Schwarzian. We shall be interested in the choice of the warp function

$$h = \left(\frac{P\zeta - Q}{R\zeta - S} \right)^s \quad (30)$$

and without loss of generality we can take $P = 1, Q = R = S = 0$ so that we can simply consider

$$h = \zeta^s \quad (31)$$

where s is a constant different from one. Note that here we could have considered a more general form $h = \zeta^s f(\zeta)$ where f is holomorphic, or $\ln |\zeta f(\zeta)|$ which is its “ $s = 0$ ” form. But this is an inessential refinement and we shall not consider it any further. From the choice of the warp function (31) we get

$$\{h; \zeta\} = -\frac{1-s^2}{4\zeta^2} \quad (32)$$

thus in the metric we have a meromorphic differential with a double pole. In this paper we shall restrict our discussion to real values of the parameter s . To exhibit the cosmic string aspect of the metric (1) with (32) we transform to new coordinates

$$\begin{aligned} u &= \frac{1}{4} \left[(s+1)^2 \rho^{s-1} U' - (s-1)^2 \rho^{-1-s} V' \right], \\ v &= \rho^{1-s} \left(V' - \rho^{2s} U' \right), \\ \zeta &= \rho e^{i\phi} \end{aligned} \quad (33)$$

where we have introduced the definitions

$$\rho = \left[\frac{R + \sqrt{R^2 + 2(s^2 - 1)U'V'}}{\sqrt{2}(s + 1)U'} \right]^{1/s}, R = (Z'\bar{Z}')^{1/2} \quad (34)$$

$$U' = \frac{t - z}{\sqrt{2}}, \quad V' = \frac{t + z}{\sqrt{2}}.$$

This brings the metric given by (1) and (32) to the form

$$ds^2 = dt^2 - dz^2 - dR^2 - s^2 R^2 d\phi^2 \quad (35)$$

which is flat space with a conical deficit determined by the parameter s . It is the metric around a straight cosmic string.

Physical properties of the metric (1) can now be summarized. Singularities of the solution (1) which are due to the necessarily-incomplete spherical wavefronts are related to conical singularities of cosmic strings as we saw above. The choice of the warp function (31) where s is a real constant different from one results in the flat metric (35) with conical deficit for $v > 0$. For $v < 0$ we have complete Minkowski space. At $v = 0$ there is an impulsive spherical wave. This is the scenario for the creation of a cosmic string with an impulsive spherical wave at its ends. The opposite scenario is physically more relevant. We start with a cosmic string, Minkowski space with conical deficit, and the string snaps at $v = 0$ with the emission of an impulsive spherical wave and Minkowski space without conical singularity is left behind. Currently acceptable values for the mass per unit length μ of a cosmic string suggest

$$s = 1 + \epsilon \quad \epsilon = G\mu \approx 10^{-6}$$

and the formulae (33) and (34) for the transformation of coordinates to Minkowski space determine the deficit angle of the conical singularity in terms of s .

In the discussion of the cosmic string above we took the constant s to be real. But there is no reason why it cannot be complex. In this case the imaginary part of s will impart rotation to the cosmic string and we get the metric of a spinning cosmic string which is Lorentz-invariant and does not violate causality, in contrast to an alternative suggestion for spinning string due to Deser, Jackiw and 't Hooft [19]. This is also an important issue which will be discussed in a separate publication [20].

Finally, it will be useful to compare the continuous metric for an impulsive pp -wave

$$ds^2 = 2 du dv - 2 \left| d\bar{\zeta} + v \theta(v) q_{\zeta\bar{\zeta}} d\zeta \right|^2 \quad (36)$$

where q is an analytic function

$$q_{\zeta\bar{\zeta}} = 0 \quad (37)$$

with that of the impulsive spherical wave given in eqs.(1) and (37). The impulsive pp -waves are obtained in limit $\lambda \rightarrow 0$ where [4]

$$\begin{aligned} u &\rightarrow \lambda^{-1}u + \lambda^{-2}, \\ v &\rightarrow \lambda v, \\ \zeta &\rightarrow \lambda^2 \zeta, \\ \{h; \zeta\} &\rightarrow \lambda^{-3} q_{\zeta\bar{\zeta}} \end{aligned} \quad (38)$$

so that we have the following correspondence

$$\begin{aligned} h = \zeta^s &\rightarrow q = \ln \zeta \bar{\zeta}, \\ h = e^{s\zeta} &\rightarrow q = \frac{1}{2} (\zeta^2 + \bar{\zeta}^2) \end{aligned} \quad (39)$$

whereby the metric of the snapping cosmic string goes over into the Aichelburg-Sexl solution [21] and the Gleiser-Pullin solution [11] corresponds to the impulsive plane wave [5].

There is, however, one important respect in which the pp -wave metric is different from the metric for spherical waves. Namely, in the case of shock waves where the Riemann tensor suffers a Heaviside step function discontinuity rather than the δ -function discontinuity characteristic of impulsive waves, it is sufficient [22] to replace $v\theta(v)$ by $v^2\theta(v)$ in the impulsive pp -wave metric of eq.(36). If we were to try the same procedure for spherical waves using eq.(1), we would find that the resulting metric fails to be an exact solution of vacuum Einstein field equations. The case of spherical shock waves requires a different treatment which has been given in [23].

4 Non-null solutions

We have seen that the metric (1) describing an impulsive spherical gravitational wave is constructed by Penrose's identification with warp of two halves

of Minkowski space with metric given by eq.(6) where $v = 0$ is a null cone. There should be a family of exact solutions of the Einstein field equations describing impulsive spherical waves of which eq.(1) is the simplest example. Such solutions will contain new parameters and the question naturally arises as to the existence of a general technique for finding them. The answer [23] is simple. We must look for metrics describing Minkowski space where $v = 0$ is again a null cone but the metric admits a set of new parameters.

The simplest such metric is obtained by transforming the coordinates in eq.(4) by

$$\begin{aligned} U' &= \frac{k}{2} v + \frac{u}{p}, \\ V' &= v + \frac{u}{p} |\zeta|^2, \\ Z' &= \frac{u}{p} \zeta \end{aligned} \tag{40}$$

where

$$p = 1 + \frac{k}{2} |\zeta|^2, \quad k = 0, \pm 1 \tag{41}$$

and we obtain the metric

$$ds^2 = 2 du dv + k dv^2 - 2 \frac{u^2}{p^2} d\zeta d\bar{\zeta} \tag{42}$$

where v is again a null coordinate and the hypersurface \mathcal{N} given by $v = 0$ is also a null cone. We shall now show that the generalization of the metric (1) which gives the impulsive spherical wave solution including the arbitrary constant k in the Robinson-Trautman solutions is obtained by Penrose's identification with warp on both sides of \mathcal{N} for the metric (42). This will also prepare the ground for the inclusion of the acceleration parameter that we shall present in the next section.

We shall again use spinor techniques and as it is evident from eqs.(40) in the non-null case the position vector of a general point on the future null cone is given by

$$x^\mu \leftrightarrow x^{AX'} = u \xi^A \xi^{X'} + v \mu^{AX'} \tag{43}$$

where we have introduced a constant second-rank spinor $\mu^{AX'}$ defined along the world-path which cannot be expressed as a bi-spinor. The vector equiv-

alent of $\mu^{AX'}$ will have magnitude proportional to k which will make it time-like, or spacelike for $k = +1, k = -1$ respectively. The explicit form of the transformation (43) is given by

$$x^{AX'} = \frac{u}{p} \begin{pmatrix} 1 & \bar{\zeta} \\ \zeta & \zeta\bar{\zeta} \end{pmatrix} + v \begin{pmatrix} \frac{1}{2}k & 0 \\ 0 & 1 \end{pmatrix} \quad (44)$$

so that the 2-component spinor ξ^A retains flagpole direction along the future null cone with vertex on the world-path which, however, is no longer null. Comparing eqs.(43) and (44) we find that the normalization conditions

$$\mu_{AX'} \xi^A \bar{\xi}^{X'} = 1 \quad (45)$$

and

$$\mu_{AX'} \mu^{AX'} = k \quad (46)$$

are satisfied.

Penrose's identification with warp is given by the arbitrary holomorphic transformation $\zeta \rightarrow h(\zeta)$. It results in the following expression for the position vector of a general point

$$\hat{x}^{AX'} = \frac{u}{\mathcal{P}} |\eta| \begin{pmatrix} 1 & \bar{h} \\ h & |h|^2 \end{pmatrix} + v \begin{pmatrix} \hat{\mu}^{00'} & \hat{\mu}^{01'} \\ \hat{\mu}^{10'} & \hat{\mu}^{11'} \end{pmatrix} \quad (47)$$

where

$$\mathcal{P} = 1 + \frac{1}{2}k |h|^2$$

and we need to determine the spinors ξ^A and $\mu^{AX'}$. The 2-component spinor ξ^A has an expression analogous to eq.(10)

$$\hat{\xi}^A = \begin{pmatrix} (\eta/\mathcal{P})^{1/2} \\ (\eta/\mathcal{P})^{1/2} h \end{pmatrix} \quad (48)$$

and the requirement of Penrose's Type *I* geometry, namely the invariance of the 1-form $\xi_A d\xi^A$ under a holomorphic mapping on the future null cone \mathcal{N} leads to

$$|\eta| = \frac{1}{|h'|} \frac{1 + \frac{k}{2} |h|^2}{1 + \frac{k}{2} |\zeta|^2} \quad (49)$$

as in eq.(11). In terms of coordinates we have

$$\begin{aligned}\hat{v} &= 0 = v, \\ \hat{u} &= \frac{u}{|h'|} \frac{1 + \frac{k}{2} |h|^2}{1 + \frac{k}{2} |\zeta|^2}, \\ \hat{\zeta} &= h(\zeta)\end{aligned}\tag{50}$$

for the identification of two halves of Minkowski space given by the metric (42) along the null cone $v = 0$. In eqs.(49) and (50) we find the first indication that will be repeated through the subject. In identification with warp $\zeta \rightarrow h(\zeta)$ is not to be taken as a mechanical dictum. It is the continuity conditions along the null cone, in this case the invariance of $\xi_A d\xi^A$, that are important and as we find in these equations both ζ and $h(\zeta)$ will appear in the final expression for the metric.

The explicit form of the second-rank spinor $\hat{\mu}^{AX'}$ is obtained from Penrose's Type *II* and *III* geometry that a holomorphic mapping with warp must preserve the normalization conditions (45) and (46) and

$$\mu_{AX'} \frac{d}{d\zeta} (\xi^A \bar{\xi}^{X'}) = 0\tag{51}$$

together with its complex conjugate that replace eqs.(21). The solution of eqs.(45), (46) and (51) gives us

$$\begin{aligned}\hat{\mu}^{00'} &= \frac{1}{4|h'|} \left(k K + p \left| \frac{h''}{h'} \right|^2 \right), \\ \hat{\mu}^{11'} &= \frac{1}{4|h'|} \left[k (|h|^2 K - 2 h \bar{h}' \bar{\zeta} - 2 \bar{h} h' \zeta) + p |m|^2 \left| \frac{h''}{h'} \right|^2 \right], \\ \hat{\mu}^{10'} &= \frac{1}{4|h'|} \left[k (h K - 2 h' \zeta) + p m \left| \frac{h''}{h'} \right|^2 \right] \\ \hat{\mu}^{01'} &= \overline{\hat{\mu}^{10'}},\end{aligned}\tag{52}$$

where

$$K = 2 + \frac{h''}{h'} \zeta + \frac{\bar{h}''}{\bar{h}'} \bar{\zeta}$$

and m is again given by eq.(22). From eqs.(47) we obtain the generalization of the transformation (24) for arbitrary values of the parameter k

$$\begin{aligned}
V' &= \frac{u}{p} \frac{|h|^2}{|h'|} + \frac{v}{4|h'|} \left[k (|h|^2 K - 2 h \bar{h}' \bar{\zeta} - 2 \bar{h} h' \zeta) + p |m|^2 \left| \frac{h''}{h'} \right|^2 \right], \\
U' &= \frac{u}{p} \frac{1}{|h'|} + \frac{v}{4|h'|} \left(k K + p \left| \frac{h''}{h'} \right|^2 \right), \\
Z' &= \frac{u}{p} \frac{h}{|h'|} + \frac{v}{4|h'|} \left[k (h K - 2 h' \zeta) + p m \left| \frac{h''}{h'} \right|^2 \right]
\end{aligned} \tag{53}$$

which for $k = +1$ is equivalent to the one given by Hogan [17] but once again spinor methods bring simplicity and compactness. From eqs.(53) we arrive at the metric for an impulsive spherical gravitational wave for arbitrary values of the parameter k [17]

$$ds^2 = 2 du dv + k dv^2 - 2 \left| \frac{u}{p} d\bar{\zeta} + p v \theta(v) \{h; \zeta\} d\zeta \right|^2 \tag{54}$$

which is continuous. It shares all the properties of the metric (1) and reduces to it in the null case $k = 0$. We note that, as we remarked earlier, besides $h(\zeta)$ also ζ enters into the final result for the metric through p given by eq.(41).

5 Accelerating solution

We have shown that metrics generalizing the impulsive spherical gravitational wave metric (1) are obtained by extending Penrose's identification with warp to two halves of Minkowski space where $v = 0$ is a null cone but the Minkowski metric contains new parameters. It was pointed out in [23] that flat metric with constant acceleration [24] is another such example. The metric for flat space with constant acceleration is obtained by the transformation

$$V' = \frac{1}{a} (1 - e^{-av}) + \frac{u}{p} e^{-av} |\zeta|^2,$$

$$\begin{aligned}
U' &= \frac{k}{2a} (e^{av} - 1) + \frac{u}{p} e^{av}, \\
Z' &= \frac{u}{p} \zeta,
\end{aligned} \tag{55}$$

where a is the acceleration parameter and p is given by (41). Applying the transformation (55) to the metric (4) we obtain

$$\begin{aligned}
ds^2 &= 2 du dv + \left(k + \frac{2au}{p} \right) \left(1 - \frac{au}{p} |\zeta|^2 \right) dv^2 \\
&\quad + \frac{2au^2}{p^2} (\bar{\zeta} d\zeta + \zeta d\bar{\zeta}) dv - 2 \frac{u^2}{p^2} d\zeta d\bar{\zeta}
\end{aligned} \tag{56}$$

which is the flat-space limit of the uniformly accelerating C -metric [24] in stereographic coordinates on the sphere and reduces to the metric (42) for $a = 0$. For the spinor description of the metric (56) it is useful to introduce new coordinates

$$\begin{aligned}
V &= \frac{1}{a} (1 - e^{-av}), \\
U &= u e^{av}, \\
Z &= \zeta e^{-av}
\end{aligned} \tag{57}$$

and the transformation (55) becomes

$$\begin{aligned}
V' &= V + \frac{U}{p} |Z|^2 \\
U' &= \frac{k}{2} \frac{V}{1 - aV} + \frac{U}{p} \\
Z' &= \frac{U}{p} Z,
\end{aligned} \tag{58}$$

where p is now given by

$$p = 1 + \frac{k}{2(1 - aV)^2} |Z|^2 \tag{59}$$

which depends on the acceleration parameter as well. In these coordinates the metric (56) assumes the form

$$ds^2 = 2 dV \left[dU + \frac{k}{2} \frac{dV}{(1 - aV)^2} + \frac{2(1 - p)}{p} \frac{aU}{1 - aV} dV \right]$$

$$-\frac{2U^2}{p^2} dZ d\bar{Z} \quad (60)$$

which has a removable singularity at $V = a^{-1}$. The hypersurface $V = 0$ is again a null cone and we can construct a new impulsive spherical wave metric by identifying with warp the two halves Minkowski space defined by $V > 0$ and $V < 0$ in the metric (60).

The position vector of a general point on the null cone of a uniformly accelerating world-path has the form

$$x^{AX'} = U \xi^A \xi^{X'} + V \mu^{AX'} \quad (61)$$

$$= \frac{U}{p} \begin{pmatrix} 1 & \bar{Z} \\ Z & |Z|^2 \end{pmatrix} + V \begin{pmatrix} \frac{k}{2(1-aV)} & 0 \\ 0 & 1 \end{pmatrix} \quad (62)$$

which follows from eqs.(58). The tangent vector to the path is given by the second rank spinor $\mu^{AX'}$. From eqs.(61) and (62) we find that the spinors ξ^A and $\mu^{AX'}$ satisfy the normalization conditions

$$\mu_{AX'} \xi^A \bar{\xi}^{X'} = F \quad (63)$$

where

$$F = \frac{1}{p} \left(1 + \frac{k}{2(1-aV)} |Z|^2 \right) \quad (64)$$

and

$$\mu_{AX'} \mu^{AX'} = \frac{k}{1-aV}. \quad (65)$$

The warp will now be given by $Z \rightarrow h(Z)$, however, we must repeat our earlier remark that it must not be taken as a dictum. In particular, the normalization condition (63) involves ζ through F , unlike the situations encountered earlier. It must remain unchanged under the warp.

Under the arbitrary holomorphic transformation of the spin-frame the position vector of a general point is given by

$$\hat{x}^{AX'} = \frac{U}{P} |\eta| \begin{pmatrix} 1 & \bar{h} \\ h & |h|^2 \end{pmatrix} + V \begin{pmatrix} \hat{\mu}^{00'} & \hat{\mu}^{01'} \\ \hat{\mu}^{10'} & \hat{\mu}^{11'} \end{pmatrix} \quad (66)$$

where

$$P = 1 + \frac{k}{2(1-aV)^2} |h|^2.$$

It follows that we may now introduce the 2-component spinor

$$\hat{\xi}^A = \begin{pmatrix} (\eta/P)^{1/2} \\ (\eta/P)^{1/2} h \end{pmatrix} \quad (67)$$

where

$$|\eta| = \frac{1}{|h'|} \frac{P}{p} \quad (68)$$

which insures that $\xi_A d\xi^A$ remains invariant under the arbitrary holomorphic transformation. Thus the identification with warp of the two halves of Minkowski space with metric (60) across $V = 0$ is given by

$$\begin{aligned} \hat{V} &= 0 = V \\ \hat{Z} &= h(Z) \\ \hat{U} &= \frac{U}{|h'|} \frac{1 + \frac{k}{2}|h|^2}{1 + \frac{k}{2}|Z|^2}. \end{aligned} \quad (69)$$

The explicit expression for the components of the second-rank spinor $\hat{\mu}^{AX'}$ follows from the requirement of Penrose's Type *II* and *III*-geometry [8]. The second fundamental form of \mathcal{N} must be the same when induced by the two flat pieces ($V < 0$) and ($V > 0$) of Minkowski space. This implies the invariance of the normalization conditions (63) and (65). From the solution of these equations we arrive at the transformation of coordinates in the region $V > 0$

$$\begin{aligned} V' &= \frac{U}{p} \frac{|h|^2}{|h'|} + \frac{V}{4|h'|} \left[q |m|^2 \left| \frac{h''}{h'} \right|^2 + \right. \\ &\quad \left. \frac{k}{1-aV} |h|^2 \left(2 + \frac{h''}{h'} \frac{m}{h} Z + \frac{\bar{h}''}{\bar{h}'} \frac{\bar{m}}{\bar{h}} \bar{Z} \right) \right] \\ U' &= \frac{U}{p} \frac{1}{|h'|} + \frac{V}{4|h'|} \left[q \left| \frac{h''}{h'} \right|^2 + \frac{k}{1-aV} \left(2 + \frac{h''}{h'} Z + \frac{\bar{h}''}{\bar{h}'} \bar{Z} \right) \right] \\ Z' &= \frac{U}{p} \frac{h}{|h'|} + \frac{V}{4|h'|} \left[q m \left| \frac{h''}{h'} \right|^2 + \right. \end{aligned} \quad (70)$$

$$\frac{k}{1-aV} h \left(2 + \frac{h''}{h'} \frac{m}{h} Z + \frac{\bar{h}''}{\bar{h}'} \bar{Z} \right) \Bigg]$$

where we have introduced

$$q = 1 + \frac{k}{2(1-aV)} |Z|^2 \quad (71)$$

and m is again given by (22).

The continuous impulsive spherical gravitational wave metric obtained by letting the vertex of the null cone to lie on a world-line with constant acceleration is given by

$$\begin{aligned} ds^2 = & 2 dV \left[dU + \frac{k}{2} \frac{dV}{(1-aV)^2} + \frac{2(1-p)}{p} \frac{aU}{1-aV} dV \right. \\ & \left. + \frac{ka}{2(1-aV)^2} V^2 \theta(V) \left[\{h; Z\} Z dZ + \{\bar{h}; \bar{Z}\} \bar{Z} d\bar{Z} \right] \right] \quad (72) \\ & - 2 \left| \frac{U}{p} d\bar{Z} + qV \theta(V) \{h; Z\} dZ \right|^2. \end{aligned}$$

It follows from the non-linear holomorphic transformation (70) of Minkowski spacetime. In successive limits $a = 0$ and $k = 0$ it reduces to (54) and (1), whereas for $k = 0$ it immediately reduces to the null case (1) because acceleration is not meaningful when the world-path is null.

6 Curvature

In order to show that the metric (72) satisfies the Einstein vacuum field equations and describes an impulsive spherical gravitational wave we shall now calculate its curvature using the Newman-Penrose formalism [25]. We start with the null co-frame

$$\begin{aligned} l &= dV, \\ n &= dU + \left[\frac{2aU(1-p)}{p} + \frac{k}{2} \frac{1}{1-aV} \right] \frac{dV}{1-aV} \end{aligned}$$

$$\begin{aligned}
& + \frac{k a}{2} \frac{V^2 \theta(V)}{(1-aV)^2} \left[\{h; Z\} Z dZ + \{\bar{h}; \bar{Z}\} \bar{Z} d\bar{Z} \right] dV, \quad (73) \\
m &= \frac{U}{p} dZ + q V \theta(V) \{\bar{h}; \bar{Z}\} d\bar{Z}
\end{aligned}$$

and the metric (72) is given by

$$ds^2 = l \otimes n + n \otimes l - m \otimes \bar{m} - \bar{m} \otimes m \quad (74)$$

the Newman-Penrose null form. Calculating the spin-coefficients with the co-frame (73) we obtain

$$\begin{aligned}
\kappa &= \tau = \pi = \epsilon = 0, \\
\rho &= -\frac{1}{\Delta} \frac{U}{p^2}, \\
\nu &= \frac{k a}{p} \frac{\bar{Z}}{(1-aV)^3}, \\
\sigma &= \frac{1}{\Delta} \frac{q}{p} V \theta(V) \{\bar{h}; \bar{Z}\}, \quad (75) \\
\gamma &= \frac{1-p}{p} \frac{a}{1-aV}, \\
\lambda &= \frac{\{h; Z\}}{\Delta} \theta(V) \left[U + \frac{k}{2p} \frac{q V}{(1-aV)^2} \right], \\
\mu &= -\frac{1}{\Delta} \left[\frac{k}{2p^2} \frac{U}{(1-aV)^2} + p q V \theta(V) \{h; Z\} \{\bar{h}; \bar{Z}\} \right], \\
\beta &= -\bar{\alpha} = \frac{k}{4p \Delta} \frac{1}{(1-aV)^2} \left[\frac{U}{p} Z + q V \theta(V) \{\bar{h}; \bar{Z}\} \bar{Z} \right]
\end{aligned}$$

where

$$\Delta = \frac{U^2}{p^2} - q^2 V^2 \theta(V) \{h; Z\} \{\bar{h}; \bar{Z}\}. \quad (76)$$

Using these spin coefficients we find that all Ricci scalars vanish identically for all values of V , while in the curvature the only nonvanishing Weyl tetrad scalar

$$\Psi_4 = -\frac{\{h; Z\}}{U} p^2 \delta(V) \quad (77)$$

suffers a Dirac δ -function discontinuity. Thus the metric (72) is of the Petrov type N exact solution of the Einstein vacuum field equations. It should be noted that the constant acceleration parameter in the metric (72) does not appear in the curvature. This is a well-known fact that is repeated for the case of uncharged metrics [24],[26].

7 Conclusion

The continuous form of impulsive gravitational wave metrics follows from Penrose's identification with warp. For impulsive spherical waves Minkowski space is cut into two pieces along a null cone which are then identified with warp, an arbitrary holomorphic transformation. Using the 2-component spinor formalism we have presented the explicit form of the holomorphic transformations that lead to the family of impulsive spherical wave metrics. For waves with smooth profile the resulting metrics are equivalent to Petrov Type N Robinson-Trautman solutions. In the continuous form of the metric for impulsive spherical waves the "wire singularities" in the Robinson-Trautman solutions reappear as Minkowski space with and without a conical deficit outside and inside the future null cone respectively. Therefore the physical interpretation of these solutions is that a snapping cosmic string is the source of the impulsive spherical wave. The full power of Penrose's method of identification with warp becomes manifest with 2-component spinor techniques and leads to the simplest derivation of the exact solutions. In this framework we have discussed the derivation known solutions and then used these tools to construct a new exact solution of the Einstein field equations that describes an impulsive spherical gravitational wave in accelerating frame.

References

- [1] Einstein A 1918 Sitzungsberichte d. Preuss. Akad. Wiss., 1. Teil, 154
- [2] Taylor J H, Wolszczan A, Damour T and Weisberg J M 1992 Nature **355** 132

- [3] This is an argument attributed to Dyson F J by Misner C W see also Misner C W, Thorne K S and Wheeler J A 1970 *Gravitation* Freeman W H and Co. San Francisco p. 978
- [4] Robinson I and Trautman J 1960 Phys. Rev. Lett. **4** 593; Proc. Roy. Soc. (London) 1962 **A 265** 463
- [5] Penrose R 1965 Rev. Mod. Phys. **37** 215
- [6] Kibble T W B 1976 J. Phys. A **9** 1398
- [7] Penrose R in *Battelle Rencontres* ed. DeWitt C and Wheeler J A (W. A. Benjamin 1968)
- [8] Penrose R in *General Relativity, Papers in Honour of Synge J L* ed. by O’Raifertaigh L (Clarendon Press, Oxford 1972)
- [9] Nutku Y and Penrose R 1992 Twistor Newsletter **34** 9
- [10] Hortaçsu M 1990 Class. Quantum Grav. **7** L165
- [11] Gleiser R and Pullin J 1989 Class. Quantum Grav. **6** L141
- [12] Griffiths J B and Podolský J 1999 Class. Quantum Grav. **16** 2937
- [13] Griffiths J B and Podolský J 2000 Annalen der Physik **9** SI59
- [14] Griffiths J B and Podolský J 2000 Class. Quantum Grav. **17** 1401
- [15] Tod K P 1999 private communication.
- [16] Penrose R and MacCallum M A H 1973 Physics Repots **6 C** 242
- [17] Hogan P A 1994 Phys. Rev. D **49** 6521
- [18] Yoshida M 1987 “Fuchsian Differential Equations” Max-Planck-Institut für Mathematik, Bonn; Vol. E11 Vieweg F & Sohn Braunschweig Wiesbaden.
- [19] Deser S, Jackiw R and ’t Hooft G 1984 Ann. Phys. (N.Y.) **152** 220
- [20] Aliev A N and Nutku Y (in preperation)

- [21] Aichelburg P C and Sexl R U 1971 Gen. Rel. and Grav. **2** 303
- [22] Szekeres P 1972 J. Math. Phys. **13** 286
- [23] Nutku Y 1992 Phys. Rev. D **44** 3164
- [24] Kinnersley W and Walker M 1970 Phys. Rev. D **2** 1359
- [25] Newman E T and Penrose R 1962 J. Math.Phys. **3** 566; 1962 J. Math.Phys. **4** 998
- [26] Plebanski J F and Demianski M 1976 Ann. Phys. (N.Y.) **98** 98